

# $f(R)$ -gravity from Killing Tensors

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We consider  $f(R)$ -gravity in a Friedmann-Lemaître-Robertson-Walker spacetime with zero spatial curvature. We apply the Killing tensors of the minisuperspace in order to specify the functional form of  $f(R)$  and the field equations to be invariant under Lie-Bäcklund transformations which are linear in the momentum (contact symmetries). Consequently, the field equations to admit quadratic conservation laws given by Noether's Theorem. We find three new integrable  $f(R)$  models, for which with the application of the conservation laws we reduce the field equations to a system of two first-order ordinary differential equations. For each model we study the evolution of the cosmological fluid. Where we find that for the one integrable model the cosmological fluid has an equation of state parameter, in which in the latter there is a linear behavior in terms of the scale factor which describes the Chevallier, Polarski and Linder (CPL) parametric dark energy model.

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## 1. INTRODUCTION

The source for the late-time cosmic acceleration [1–5] has been attributed to an unidentified type of matter with a negative parameter in the equation of state, the dark energy. The cosmological constant,  $\Lambda$ , leading to the  $\Lambda$ -cosmology, is the simplest candidate for the dark energy. In  $\Lambda$ -cosmology the universe consists of two perfect fluids, namely, the dust fluid (dark matter) with zero pressure, and the dark energy fluid which corresponds to the cosmological constant  $\Lambda$ , with parameter,  $w_\Lambda = -1$  in the equation of state. The terms which correspond to the cosmological constant in the field equations can be seen in two ways, as a cosmic fluid with constant energy density and constant negative parameter in the equation of state, or as an additional component which follows from the modification of the Einstein-Hilbert Action in General Relativity. However,  $\Lambda$ -cosmology suffers from two major problems, the fine tuning and the coincidence problems [6–8].

In recent years other cosmological models have been introduced in order to explain the acceleration phase of the universe. Some of them introduce a cosmic fluid into Einstein's General Relativity [9–17], while some other models modify the Einstein-Hilbert Action [18–20, 22–24].

In this work we are interested in the  $f(R)$ -gravity in a Friedmann-Lemaître-Robertson-Walker spacetime (FLRW) with zero spatial curvature.  $f(R)$ -gravity is a modified theory of gravity in which the Action Integral of the field equations is a function  $f$ , of the Ricci scalar  $R$  of the underlying geometry [19] (for review see [20, 21]). In the case for which  $f(R)$  is a linear function,  $f(R)$ -gravity reduces to standard General Relativity, with or without the term due to the cosmological constant. The functional form of  $f(R)$  is still unknown and different forms for  $f(R)$  provide us with different dynamics, i.e. evolution of the universe. The well-known Starobinsky model, with  $f(R) = R + \alpha R^2$ , has been proposed as an inflationary model of gravity [25]. A class of viable models which describe the accelerated expansion of the universe can be found in [26]. Other models which have been proposed in the literature can be found in [27–35] and references cited therein, while some cosmological data analysis of  $f(R)$  models can be found in [36, 37].

The purpose of this paper is the determination of the functional form of  $f(R)$  in order for the modified field equations to admit quadratic conservation laws. To perform this analysis we use the method of group invariant transformations, specifically the selection rule that we assume is that the field equations are invariant under a contact transformation which are defining as one-parameter transformations in the tangent bundle of the dynamical system.

According to the well-known Noether's Theorems, the existence of a group invariant transformation in the field equations is equivalent to the existence of conservation flow. Noether's theorem states that for every one-parameter transformation of the Action Integral of a Lagrangian function which transforms the Action Integral in such a way that the Euler-Lagrange equations are invariant. A conservation flow corresponds to the transformation. Contact transformations are important in physical science as they are related with important conservation laws such as the Runge-Lenz Vector, the Lewis Invariant; contact transformations provide also the conservation law in the MICZ-Problem [58, 59].

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Group invariant transformations cover a range of applications in gravitational physics and cosmology. For instance, Lie point symmetries have been used for the determination of closed-form solutions in a model with charged perfect fluids in spherically symmetric spacetimes [38, 39]. On the other hand, Noether point symmetries have been introduced as a selection rule for the determination of the functional form of the potential in scalar field cosmology in [40]. Since then that method has been applied in various cosmological models and new solutions have been found (for instance see [41–50] and references cited therein). In [51] Noether symmetries were applied in the scalar field cosmological scenario as a geometric selection rule for the determination of the functional form of the potential. In theories with minisuperspace, Noether point symmetries are generated by the collineations of the minisuperspace. Collineations are the generators of one-parameter transformations which transform geometric objects under a certain rule [52]. However, the minisuperspace is defined by the cosmological model and the existence of collineations depends upon the model. Hence the requirement of the existence of a Noether point symmetry in the cosmological Lagrangian is also a self-criterion because we let the theory select the functional form of the model. That selection rule is consistent with the geometric character of gravity. This geometric approach has been applied in various cosmological models and new integrable cosmologically viable models have arisen [53–55].

The application of Noether (point) symmetries which are the generators of one-parameter point transformation for the  $f(R)$ -cosmology in an FLRW spacetime, in which the spacetime comprises a perfect fluid with zero pressure, has been performed before in [56]. Recently the same analysis has been performed and for a general perfect fluid with nonzero equation of state parameter [57], whereas for some locally rotational spacetimes the Noether point symmetry classification of  $f(R)$ -cosmology can be found in [60]. Some other  $f(R)$ -models with closed-form solutions can be found in [61], while the application of point transformations in  $f(R)$ -gravity in static spherically symmetric spacetimes can be found in [62, 63].

Another geometric selection rule which is based upon the group invariant transformations of the Wheeler-DeWitt equations was introduced in [64]. It has been shown that the existence of a Lie point symmetry for the Wheeler-DeWitt equation is equivalent with the existence of a Noetherian conservation law for the classical field equations. The main result was applied in a scalar field cosmological model with a perfect fluid for which a new integrable scalar field model was derived which it has been shown that the model provide us with a viable inflationary scenario [65].

The application of contact symmetries in cosmological studies is not new. Contact symmetries have been applied for the determination of conservation laws in various models in [41]. In [66] contact symmetries have been applied for the determination of the potential in scalar field cosmology. As in the case of point symmetries, contact symmetries (or Dynamical Noether symmetries) are also a geometric selection rule because they follow from a class of collineations of the minisuperspace which are called Killing tensors. Following the method which was presented in [66] we apply the same selection rule for the determination of the unknown functional form,  $f$ , in modified  $f(R)$ -cosmology in the metric formalism. The plan of the paper is as follows.

In Section 2 we present the field equations in  $f(R)$ -gravity and we define our model which is a spatially flat FLRW spacetime and contains a dustlike fluid which is not interacting with gravity in the Action Integral. The basic properties of the Lie-Bäcklund symmetries and Noether's Theorem for contact transformations are given in Section 3. In Section 4, we use the Killing tensors of the minisuperspace in order to determine the  $f(R)$ -models in which the field equations are invariant under contact transformations. For each model we give the corresponding quadratic conservation law. We find five models which admit quadratic conservation laws. Two of these have been found from the application of Noether's Theorem for one-parameter point transformations.

In Section 5 we apply the extra conservation laws in order to reduce the order the dynamical system, which is defined by the field equations. For the three new integrable models we show that two of the models are supported by Lie surfaces while the third model is supported by a Liouville surface. For each of the models we study the evolution of the equation of state parameter for the cosmological fluid which corresponds to  $f(R)$ -gravity. Appendix A completes our results where we present the quadratic conservation laws of  $f(R)$ -gravity in a spatially nonflat FLRW spacetime. Finally in Section 6, we draw our conclusions.

## 2. FIELD EQUATIONS IN $f(R)$ -GRAVITY

We consider a FLRW universe with zero spatial curvature in which the fundamental line element is given by the following expression

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2). \quad (1)$$

The line element (1) describes an isotropic universe and admits as Killing algebra the Killing vectors (KVs) of the three-dimensional Euclidean space, that is, the  $T_3 \otimes_s SO(3)$  Lie algebra consisting of the three translations and the three rotation symmetries of  $E^3$ .

For the gravitation Action Integral we consider that of  $f(R)$ -gravity, that is,

$$S = \int dx^4 \sqrt{-g} \frac{1}{2k} f(R) + S_m, \quad (2)$$

where  $S_m = \int dx^4 \sqrt{-g} L_m$ , corresponds to the matter term,  $R$  is the Ricci scalar of the underlying geometry and  $k = 8\pi G$ . We assume that  $S_m$  describes a dustlike fluid minimally coupled to gravity

Variation of the Action Integral (2) with respect to the metric leads to the following field equations [20]

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\sigma \nabla^\sigma) f' = k T_{\mu\nu}, \quad (3)$$

where prime,  $f'(R)$ , denotes total differentiation with respect to  $R$ , and  $\nabla_\mu$  is the covariant derivative associated with the Levi-Civita connection of the underlying Riemannian space with metric tensor  $g_{\mu\nu}$ . Furthermore  $R_{\mu\nu}$  is the Ricci tensor of  $g_{\mu\nu}$  and  $T_{\mu\nu}$  is the energy-momentum tensor for the matter component.

Furthermore the energy-momentum tensor,  $T_{\mu\nu}$ , satisfies the Bianchi identity  $\nabla^\nu T_{\mu\nu} = 0$ . From the field equations (3) we observe that in the case for which  $f(R)$  is a linear function standard General Relativity is fully recovered.

In the context of the FLRW spacetime (1), in which the Ricci scalar is<sup>1</sup>

$$R = 6 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right) \quad (4)$$

and for a dustlike fluid with energy-momentum tensor  $T_{\mu\nu} = \rho_m u_\mu u_\nu$ , where  $u_\mu = \delta_\mu^t$ , is the comoving observer,  $u^\mu u_\mu = -1$ . Furthermore, from the Bianchi identity for the tensor,  $T_{\mu\nu}$ , we derive the conservation law

$$\dot{\rho}_m + 3H\rho_m = 0, \quad (5)$$

the solution of which is  $\rho_m = \rho_{m0} a^{-3}$ , where  $\rho_{m0}$  is the energy density for the dustlike fluid at the present time.

Hence from (3) we derive the following (modified) Friedmann's equations

$$3f'H^2 = k\rho_m + \frac{f'R - f}{2} - 3Hf''\dot{R}, \quad (6)$$

and

$$2f'\dot{H} + 3f'H^2 = -2Hf''\dot{R} - (f'''\dot{R}^2 + f''\ddot{R}) - \frac{f - Rf'}{2}, \quad (7)$$

where  $H = \dot{a}/a$  is the Hubble parameter.

Equations (6), (7) can be written as follows

$$3H^2 = k_{eff}(\rho_m + \rho_f), \quad (8)$$

and

$$2\dot{H} + 3H^2 = -k_{eff}p_f, \quad (9)$$

where  $k_{eff} = k(f')^{-1}$  is the effective gravitational parameter and  $\rho_f$ ,  $p_f$  are the fluid components of  $f(R)$  gravity, that is,

$$\rho_f = \frac{f'R - f}{2} - 3Hf''\dot{R} \quad (10)$$

and

$$p_f = 2Hf''\dot{R} + (f'''\dot{R}^2 + f''\ddot{R}) + \frac{f - Rf'}{2}. \quad (11)$$

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<sup>1</sup> Overdot denotes total differentiation with respect to “ $t$ ”.

Therefore the parameter of the equation of state (EoS) for the fluid components of  $f(R)$ -gravity,  $w_f = p_f/\rho_f$ , has the following expression

$$w_f = \frac{p_f}{\rho_f} = - \frac{(f - Rf') + 4Hf''\dot{R} + 2(f'''\dot{R}^2 + f''\ddot{R})}{(f - Rf') + 6Hf''\dot{R}}, \quad (12)$$

from where we can see that, when  $f(R) = R - 2\Lambda$ , expression (12) gives  $w_f = -1$ . In order for  $k_{eff}$  be a positive function,  $f' > 0$ , should hold. This is also required in order for the final attractor of the field equations to be a de Sitter point (for details see [35]). Furthermore, condition  $f' > 0$  is necessary in order to avoid the existence of ghosts. Another important constrain in which the  $f(R)$  function should satisfies is  $f'' > 0$ , for  $R \geq R_0$ , where  $R_0$ , is the Ricci scalar today, in order the theory to be consistency with local gravity tests. On the other hand, the violation of the later constrain introduce tachyonic instability, and a nonwell defined post-Newtonian limit [67, 68]. Furthermore, from the solar system tests we have that  $f(R) \approx R - 2\Lambda$ , which means that  $f(R)$  should reduced to General Relativity. Finally, if we assume that the at the late-time the model has a stable de Sitter behavior then the following condition should be satisfied,  $0 < \frac{Rf''}{f'}(r) < 1$  at  $r = -\frac{Rf'}{f} = -2$  [35].

## 2.1. Lagrange multiplier and minisuperspace

In contrast to General Relativity, which is a second-order theory,  $f(R)$ -gravity is a fourth-order theory. This can be seen by substituting the Ricci scalar (4) into (7). Another way to derive the modified Friedmann's equations (6), (7) and (4), is with the use of a Lagrange multiplier<sup>2</sup> [69]. Lagrange multipliers are useful to reduce the order of the differential equations. However, at the same time the dimension of the space of the dependent variables is increased.

With the use of a Lagrange multiplier  $\lambda$ , in the gravitational action integral (2), we have

$$S = \frac{1}{2k} \int dx^4 \sqrt{-g} \left[ f(R) - \lambda \left( R - 6 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right) \right) \right] + \int dx^4 \sqrt{-g} \rho_{m0} a^{-3} \quad (13)$$

where we have used equation (4) and the solution of the Bianchi identity (5) for the perfect fluid. Furthermore, the condition  $\frac{\partial S}{\partial R} = 0$ , gives  $\lambda = f'(R)$ .

Hence we find that the Lagrangian of the modified Friedmann's equations is

$$L(a, \dot{a}, R, \dot{R}) = 6af'\dot{a}^2 + 6a^2f''\dot{a}\dot{R} + a^3(f'R - f) + \rho_{m0}. \quad (14)$$

Therefore, the field equations can be seen as the Euler-Lagrange equations of (14) with respect to the variables  $\{a, R\}$  and the first modified Friedmann's equation (6) can be seen as the Hamiltonian constraint<sup>3</sup> of the dynamical system with Lagrangian (14), that is,

$$\mathcal{E} = 6af'\dot{a}^2 + 6a^2f''\dot{a}\dot{R} - a^3(f'R - f), \quad (15)$$

where the constant  $\mathcal{E}$  is related to  $\rho_{m0}$  as follows,  $\mathcal{E} = 2k\rho_{m0}$  or  $\mathcal{E} = 6\Omega_m H_0^2$ . In the last expression  $\Omega_m = k\rho_{m0}/(3H_0^2)$  and  $H_0$  is the Hubble constant.

We observe that Lagrangian (14), describes the motion of a particle in the space of variables  $\{a, R\}$ , and it is in the form  $L = K + U$ , where  $K$  is the Kinetic energy which defines the minisuperspace with line element

$$ds^2 = 12af'da^2 + 6a^2f''dadR, \quad (16)$$

and  $U = a^3(f'R - f) + \rho_{m0}$ , is the effective potential.

On the other hand, the field equations (3) describe the evolution of the scale factor  $a(t)$  in a fourth-order theory. The latter means that the introduction of the Lagrange multiplier  $\lambda$ , in the gravitational action integral, reduced the order of the theory to that of a second-order theory whereas in the same time the degrees of freedom increased. For an extensive discussion on the degrees of freedom in modified theories of gravity see [75].

<sup>2</sup> For applications of the Lagrange multipliers in high-order theories of gravity see [70–73].

<sup>3</sup> Alternatively equation (6) can be derived from the Euler-Lagrange equation with respect to a new variable,  $N$ , which arises from the lapsed time  $dt = Nd\tau$  [74].

In the following sections we discuss the application of contact transformations of differential equations. We apply as a selection rule for the determination of the unknown function  $f(R)$  the requirement that the Action Integral of the field equations with Lagrangian (14) be invariant under contact transformations, equivalently the dynamical system to admit dynamical Noether symmetries, consequently, quadratic conservation laws. Below we assume that  $f''(R) \neq 0$ , otherwise Lagrangian (14) is that of General Relativity.

### 3. KILLING TENSORS AND NOETHER'S THEOREM

By definition a vector field  $X$  is called a Lie-Bäcklund symmetry of a second-order differential equation  $\Xi(t, x^k, \dot{x}^k, \ddot{x}^k) = 0$ , when  $X$  is the generator of the infinitesimal transformation [76, 77],

$$t' = t + \varepsilon \xi(t, x^k, \dot{x}^k), \quad (17)$$

$$x'^i = x^i + \varepsilon \eta^i(t, x^k, \dot{x}^k), \quad (18)$$

that is,  $X = \frac{\partial t'}{\partial \varepsilon} \partial_t + \frac{\partial x'^i}{\partial \varepsilon} \partial_i$ , which leaves invariant the differential equation  $\Xi$ , i.e.,  $\Xi(t', x'^k, \dot{x}'^k, \ddot{x}'^k) = 0$ , or  $X^{[2]}(\Xi) = 0$ , where  $X^{[2]}$  is the second prolongation of  $X$ , [78].

Infinitesimal transformations of special interest are the (Lie) point transformations, i.e.,  $\frac{\partial \xi}{\partial \dot{x}^k} = \frac{\partial \eta^i}{\partial \dot{x}^k} = 0$ , and the contact transformations in which  $\xi, \eta$  are linear functions of the first derivatives of the dependent variables. For the Lie-Bäcklund transformations (except the point transformations) it has been shown that transformation (17), (18), is equivalent to the transformation [78, 79]

$$x'^i = x^i + \varepsilon \zeta(t, x^k, \dot{x}^k), \quad (19)$$

with generator  $\bar{X} = \zeta(t, x^k, \dot{x}^k) \partial_i$ . Transformation (19) is called the canonical transformation of (17), (18) and  $\bar{X}$  is the canonical form of  $X$ .

In this work we are interested in contact transformations for which the generator,  $\bar{X}$ , has the following form

$$\bar{X} = K_j^i(t, x^k) \dot{x}^j \partial_i, \quad (20)$$

where the second-rank tensor  $K_{ij}$  is symmetric on the indices, that is,  $K_{[ij]} = 0$ .

For differential equations which arise from a variational principle there are the two well-known Noether's Theorems [80]. The first theorem relates the action of the transformation (17), (18) on the Action Integral for the Lagrangian  $L = L(t, x^k, \dot{x}^k)$  of the differential equations,  $\Xi$ , in order for the latter to be invariant. Specifically, if there exists a function  $\sigma$  such that

$$X^{[1]}L + L\dot{\xi} = \dot{\sigma}, \quad (21)$$

then the Euler-Lagrange equations of Lagrangian  $L$  are invariant under the action of the transformation (17), (18) and the generator  $X$  is called a Noether symmetry.  $X^{[1]}$ , denotes the first prolongation of  $X$ .

Condition (21) is that which was originally introduced by Emmy Noether in her originally paper [80]. Incorrectly it is termed as Noether Gauge symmetry condition [81–83]. Function  $\sigma$  is not a gauge function, but a boundary term introduced to allow for the infinitesimal changes in the value of the Action Integral produced by the infinitesimal change in the boundary of the domain caused by the infinitesimal transformation of the variables in the action integral.

The second Noether's Theorem relates the existence of Noether symmetries with that of conservation laws. Hence, if  $X$  is the generator of the infinitesimal transformation (17), (18) which satisfies the symmetry condition (21) for a specific function  $\sigma$ , then the function

$$I = \xi \left( \frac{\partial L}{\partial \dot{x}^k} \dot{x}^k - L \right) - \eta^i \frac{\partial L}{\partial \dot{x}^i} + \sigma, \quad (22)$$

is a conservation law for the dynamical system with Lagrangian  $L$ . However, for nonpoint transformations we can always apply the canonical transformation (20) and condition (21) takes the simple form

$$\bar{X}^{[1]}L = \dot{\sigma}. \quad (23)$$

As we discussed above, Lagrangian (14), describes an autonomous dynamical system and it is in the form

$$L(x^i, x^j) = \frac{1}{2} \gamma_{ij} \dot{x}^i \dot{x}^j + V_{eff}(x^k), \quad (24)$$

where,  $\gamma_{ij}$ , is the minisuperspace of the field equations with line element (16), or equivalently,

$$ds^2_{(\gamma)} = 12a\phi da + 12a^2 d\phi^2, \quad (25)$$

and effective potential,  $V_{eff}(x^k) = a^3 V(\phi) + \rho_{m0}$ , where now the new variable  $\phi$ , is  $\phi = f'(R)$ , and

$$V(\phi) = (f'R - f). \quad (26)$$

We remark that in the coordinates  $\{a, \phi\}$  the Lagrangian, (14), is that of a Brans-Dicke scalar field with zero Brans-Dicke parameter [20, 67] which is also called O'Hanlon massive dilaton gravity [90]. Note that equation (26) is the first-order Clairaut equation.

For the space of the dependent variables  $\{a, \phi\}$ , we have that the general form of the contact symmetry (20) has the following form

$$\begin{aligned} \bar{X} = & \left( K^a_a(t, a, \phi) \dot{a} + K^a_\phi(t, a, \phi) \dot{\phi} \right) \partial_a + \\ & + \left( K^\phi_a(t, a, \phi) \dot{a} + K^\phi_\phi(t, a, \phi) \dot{\phi} \right) \partial_\phi \end{aligned} \quad (27)$$

The contact transformations with generator (20), which are Noether symmetries<sup>4</sup> for Lagrange functions of the form (24), have been studied previously in the literature and it has been shown that the tensor field  $K^i_j$  of (20) is time-independent and is a Killing Tensor for the metric  $\gamma_{ij}$ , that is,  $[\mathbf{K}, \gamma]_{SN} = K_{(ij;k)} = 0$ , where  $[\cdot]_{SN}$ , denotes the Schouten-Nijenhuis Bracket<sup>5</sup> and the following condition holds,

$$K_{(i}{}^j V_{eff,j)} + \sigma_{,i} = 0, \quad (28)$$

in which  $\sigma = \sigma(x^k)$ , [84–86]. Furthermore, from (22) it follows that the corresponding Noether conservation law (22), is time-independent and quadratic in the momentum.

For the space (25), the vector field (27) is time-independent, that is,  $(K^a_a)_{,t} = (K^a_\phi)_{,t} = (K^\phi_\phi)_{,t} = 0$ , and  $K_{ij}$ , is a Killing Tensor of the minisuperspace (25). Furthermore condition (28) gives the following system

$$K^a_a(aV_{,\phi}) + (3V_\phi - 2\phi V_{,\phi}) K^a_\phi + a^{-2} \sigma_\phi = 0, \quad (29)$$

$$V_{,\phi} (aK^\phi_a - 2\phi K^\phi_\phi) + 3VK^\phi_\phi + a^{-2} \sigma_{,a} - 2a^{-3} \phi \sigma_{,\phi} = 0. \quad (30)$$

Hence in order to solve the system (29), (30) and determine the functions  $V(\phi)$ , in which the modified field equations admit quadratic conservation laws, we have to find the Killing Tensors (KTs)  $K_{ij}$ , of the two-dimensional metric (25). It is easy to see that the Ricci scalar of (25) vanishes which means that  $\gamma_{ij}$  is a flat space. However, the signature of  $\gamma_{ij}$  is Lorentzian, that is, the Lagrangian, (28), describes the motion of a particle in the  $M^2$  space. Moreover, because  $\gamma_{ij}$  is the flat space, the Killing Tensors are reducible [91, 92]. The latter gives that the KT's are constructed by the tensor product of the Killing vectors (KV's) of  $\gamma_{ij}$ . The Killing vectors and the Killing tensors of the two-dimensional space (26) are given in Appendix B.

In the following we give the form of the potential  $V(\phi)$ , the corresponding function  $f(R)$ , and the quadratic conservation law which follow from the symmetry conditions (29), (30).

#### 4. $f(R)$ -GRAVITY WITH QUADRATIC CONSERVATION LAWS

The requirement of the existence of group invariant transformations, symmetries, on theories of gravity is twofold; symmetries can be used as a geometric selection rule to constrain the unknown parameters of the models and to derive new analytical solutions. Furthermore this selection rule is consistent with the geometric character of gravitational

<sup>4</sup> Usually these transformations are called dynamical Noether symmetries.

<sup>5</sup> For some applications of the Killing tensors in General Relativity see for instance [87–89].

theories. The reason is that the group invariant transformation of the field equations is related with the geometry which the dependent variables define, in our consideration with the minisuperspace (25). There exists a unique connection between the collineations of the minisuperspace and the symmetries of the field equations. In particular Noether point symmetries are related with the elements of the conformal algebra of the minisuperspace [51], whereas as we discussed before contact symmetries are generated by the KT's of the minisuperspace. Hence, in such a way, by using as a selection rule the existence of group invariant transformation in the field equations we let the theory select the corresponding model.

On the other hand, analytical solutions are important in order to understand the evolution of the universe. For instance, the field equations in  $\Lambda$ -cosmology are maximally symmetric and are invariant under the same group of invariant transformations with that of the linear second-order differential equation<sup>6</sup>. Therefore symmetries can be used in order to recognize, or define, well-known systems in gravity.

In our consideration the solution of the system (29), (30), provides us with the function form of  $V(\phi)$ , consequently the  $f(R)$ -theory, in which the field equations admit quadratic conservation laws. The existence of a conservation law, which is in involution and independent with the Hamiltonian, for the field equations which Lagrangian (14) defines the evolution of the system in the phase-space, that is, the field equations form an integrable dynamical system.

From the system (29), (30), we find that the only KT which produces a contact symmetry for arbitrary function  $V(\phi)$ , is the metric tensor  $\gamma_{ij}$ . The contact symmetry is the Hamiltonian flow and the corresponding Noetherian conservation law is the Hamiltonian (15). However, for specific  $V(\phi)$ , i.e.  $f(R)$  function, the Lagrangian (14), is invariant under additional contact symmetries. We have the following cases<sup>7</sup>.

(I) When  $V_I(\phi) = V_1\phi + V_2\phi^3$ , the field equations admit the quadratic conservation law

$$I_I = 3 \left( \phi \dot{a} + a \dot{\phi} \right)^2 - V_1 a^2 \phi^2 \quad (31)$$

generated by the KT  $K_{22}^{ij}$ .

(II) If  $V_{II}(\phi) = V_1\phi - V_2\phi^{-7}$ , the field equations admit the Noetherian conservation law

$$I_{II} = 3a^4 \left( \phi \dot{a} - a \dot{\phi} \right)^2 + 4V_2 a^6 \phi^{-6}, \quad (32)$$

which follows from  $K_{11}^{ij}$ .

(III) For  $V_{III}(\phi) = V_1 - V_2\phi^{-\frac{1}{2}}$ , the KT  $K_{13}^{ij}$  generates a contact transformation for the field equations in which the corresponding conservation law is

$$I_{III} = 6a^3 \dot{a} \left( a \dot{\phi} - \phi \dot{a} \right) - a^5 \left( \frac{3}{5} V_1 - V_2 \phi^{-\frac{1}{2}} \right). \quad (33)$$

(IV) When  $V_{IV}(\phi) = V_1\phi^3 + V_2\phi^4$ , the contact symmetry of the field equations follows from the KT  $K_{12}^{ij}$ , which produce the conservation law

$$I_{IV} = 12a^2 \left( a^2 \dot{\phi}^2 - \phi^2 \dot{a}^2 \right) + (a\phi)^4 (3V_1 + 4V_2\phi). \quad (34)$$

(V) Finally, for  $V_V(\phi) = V_1(\phi^3 + \beta\phi) + V_2(\phi^4 + 6\beta\phi^2 + \beta^2)$ , the field equations admit the Noetherian conservation law

$$I_V = 12a^2 \left[ (\beta - \phi^2) \dot{a}^2 + a^2 \dot{\phi}^2 \right] + a^4 (\beta - \phi^2) [V_1 (\beta + 3\phi^2) + 4V_2 (3\beta\phi + \phi^3)], \quad (35)$$

which follows from the linear combination of the two Killing tensors,  $K_{12}^{ij} + \beta K_{33}^{ij}$ .

We continue with the solution of the Clairaut equation, (26), which provides us with the corresponding  $f(R)$  functions.

<sup>6</sup> Specifically the dynamical system of  $\Lambda$ -cosmology is that of the “hyperbolic oscillator”.

<sup>7</sup> Recall that we consider  $f''(R) \neq 0$ , that is,  $V_{,\phi\phi} \neq 0$ .

#### 4.1. $f(R)$ -models

In order to determine the form of function  $f(R)$ , from the potentials  $V_I - V_V$  we have to solve the Clairaut equation, (26). The Clairaut equation always admits the linear solution,  $f(R) = f_0 R + f_1$ , and a singular solution is given by the differential equation,  $V_\phi - R = 0$  [93]. In our consideration we are interested in the singular solution which provides us with the functional form of  $f(R)$  for each potential.

Hence, for  $V_I(\phi)$ , we find that the corresponding  $f(R)$  function is

$$f_I(R) = f_0^I (R - V_1)^{\frac{3}{2}}, \quad (36)$$

in which  $f_0 = \frac{2\sqrt{3}}{9\sqrt{V_2}}$ .

Moreover from  $V_{II}(\phi)$  we have

$$f_{II}(R) = f_0^{II} (R - V_1)^{\frac{7}{8}}, \quad (37)$$

where  $f_0^{II} = \frac{8}{7} (7V_2)^{\frac{7}{8}}$ .

For  $V_{III}(\phi)$  the corresponding  $f(R)$  function is

$$f_{III}(R) = f_0^{III} R^{\frac{1}{3}} - V_1, \quad (38)$$

where  $f_0^{III} = 3 (2^{-1} V_2)^{\frac{2}{3}}$ .

The  $f(R)$  function which corresponds to the potential  $V_{IV}(\phi)$  is given by

$$f_{IV}(R) = \frac{1}{4V_2} \int \left[ \frac{(F(R))^2 + V_1^2}{F(R)} - V_1 \right] dR, \quad (39)$$

where

$$(F(R))^3 = 8V_2^2 R - V_1^3 + 4V_2 \sqrt{4V_2^2 R^2 - RV_1^3}. \quad (40)$$

For  $V_2 = 0$ , (39) gives  $f_{IV}^1(R) = \frac{2\sqrt{3}}{9\sqrt{V_2}} R^{\frac{3}{2}}$ , whereas for  $V_1 = 0$  we have the closed-form solution,  $f_{IV}^2(R) = \frac{3}{8} (2V_2^{-1})^{1/3} R^4$ . Furthermore from (40) we have that  $\text{Im} F^3(R) = 0$  when  $R(4V_2^2 R - V_1^3) > 0$ . Moreover, for  $R \gg V_1^3/V_2^2$ , we have that  $F(R) \simeq g_1 R^{1/3}$ . Hence from (39) we find that

$$f_{IV}^3(R) \simeq \frac{V_1}{4V_2} R + \frac{3}{16V_2 g_1} \left( 2V_1 (g_1 R)^{2/3} + (g_1 R)^{4/3} \right). \quad (41)$$

However, for  $R \ll V_1^3/V_2^2$ ,  $F(R)$  is constant which gives the limit of General Relativity.

Finally from  $V_V(\phi)$  we have that

$$f_V(R) = \frac{1}{4V_2} \int \left[ \frac{(\bar{F}(R))^2 + \Sigma_0}{\bar{F}(R)} - V_1 \right] dR. \quad (42)$$

In the last expression  $\Sigma_0 = (V_1^2 - 16\beta V_2^2)$  and the function  $\bar{F}(R)$  is given by

$$\begin{aligned} (\bar{F}(R))^2 &= 8V_2^2 R - V_1 \Sigma_0 + \\ &+ 4V_2 \sqrt{4V_2^2 R^2 - V_1 R \Sigma_0 + \beta \Sigma_0^2}. \end{aligned} \quad (43)$$

We can see that again, for  $R \gg (V_1^2 - 16\beta V_2^2)/V_2^2$ , function  $f_V(R)$  is given by (41).

In the case for which  $\Sigma_0 = 0$ , i.e.,  $V_1 = \pm 4V_2\sqrt{\beta}$ , the potential  $V_V(\phi)$  becomes  $V_V(\phi) \simeq (\phi \pm \sqrt{\beta})^4$ , whereas from expressions (42) and (26) we derive that

$$f_V^1(R) = \mp \sqrt{\beta} R + f_1 R^{\frac{4}{3}}, \quad (44)$$

in which the new constant is  $f_1 = 3 (2^{-1} V_2)^{\frac{8}{3}}$ .



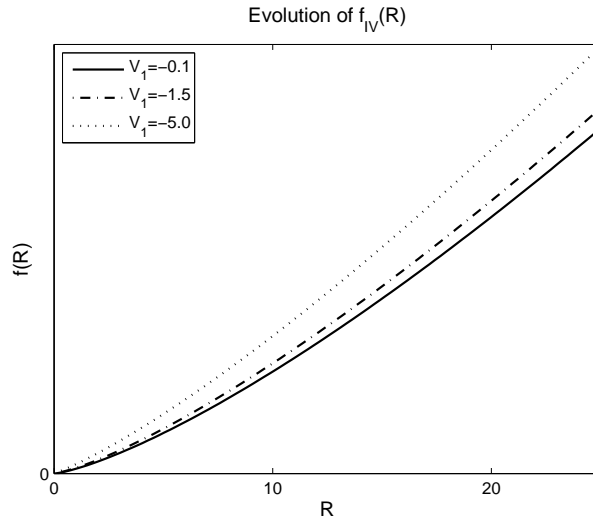


FIG. 1: Evolution of the function  $f_{IV}(R)$ , (39), for different values of the constant  $V_1$  in the range  $R \in [0, 25]$ . For the plot we select  $V_2 = 1$  with  $f_{IV}(0) \simeq 0$ . The solid line is for  $V_1 = -0.1$ , the dash-dot line is for  $V_1 = -1.5$  whereas the dot-dot line is for  $V_1 = -5.0$ . From the plot it is easy to see that holds  $f'_{IV} > 0$ , and  $f''_{IV} > 0$ .

Furthermore, the quadratic conservation laws,  $I_{I-V}$ , are independent with the Hamiltonian (15) and it holds that  $\{I_{I-V}, \mathcal{E}\} = 0$ , where  $\{, \}$  is the Poisson Bracket. Hence the models,  $f_{I-V}$ , are integrable.

In section 2 we discussed some conditions in which the  $f(R)$ -theory should satisfy in order to be viable. It is easy to see that the analytical  $f(R)$ -models,  $f_I$ ,  $f_{II}$ ,  $f_{III}$ ,  $f_{IV}^3$  and  $f_V^1$ , satisfy the condition  $f' > 0$ , which indicates that the theories are ghost free. As far as concerns the stability condition  $f'' > 0$ , the models  $f_{II}$ , and  $f_{III}$  violate the condition, hence tachyonic instability presented.

However another important constrain on the  $f(R)$ -models is that at the limit  $R \rightarrow 0$ , the theory should has a similar behavior with that of General Relativity. From the above models only the  $f_V^1(R)$  model gives that  $f(R \rightarrow 0) \simeq R$ , while for the  $f_{IV}^3$  model holds  $f_{IV}^3(R \rightarrow 0) \simeq R^{2/3}$ , which means that the model is not consistence with the local gravity tests.

In Fig. 1, we give the evolution of  $f_{IV}(R)$  for different values of  $V_1$  in the same range for  $R$ . From the figure we observe that, when  $V_1$  increases, the evolution of  $f_{IV}(R)$  is almost linear. Specifically the dot-dot line, which is for  $V_2 = 1$ ,  $V_1 = -5$ ,  $R \in [0, 25]$ , is approximated very well by the quadratic polynomial  $f(R) = a_1 R + a_2 R^2 + a_0$ , where the constants,  $a_{1-3}$ , are  $a_1 \simeq 3.7$ ,  $a_2 \simeq 8 \cdot 10^{-3}$  and  $a_3 \simeq -2.5 \cdot 10^{-2}$ . For higher-order polynomials of the form  $f(R) = \sum_{K=0} a_K R^K$ , we find that for  $K > 2$ ,  $|a_K| \lesssim 10^{-5}$ . Furthermore we observe that  $f'_{IV} > 0$ ,  $f''_{IV} > 0$ , which indicates that model is ghost free. We note that the same results hold for the  $f_V(R)$  model.

The  $f_I(R)$  and  $f_{II}(R)$  models for  $V_1 = 0$ , or  $V_1 \neq 0$ , are not new and have previously been found in [56] from the application of Noether's Theorem for point transformations to the Lagrangian, (14), of the field equations, and belong to the family of models  $f(R) = (R^b - 2\Lambda)^c$  [29]. The field equations for those models admit Noether point symmetries which form<sup>8</sup> the  $A_3$  and the  $A_{3,8}$  (or  $sl(2, R)$ ) Lie algebras, respectively. Moreover  $f_{II}(R)$  model is the Ermakov-Pinney system, in  $M^2$ , and the quadratic conservation law, (32), is the Ermakov-Lewis invariant. The closed-form solutions of  $f_I(R)$  and  $f_{II}(R)$  models can be found in [56, 98]. For both models, for  $V_1 = 0$  the closed-form solution for the scale factor is a power law whereas for  $V_1 \neq 0$  the scale factor has exponential expansion in which the late-time solution describes the de Sitter Universe.

<sup>8</sup> In the Mubarakzyanov Classification Scheme [94–97]

## 4.2. Existence of de Sitter solutions

In [99], it has been shown that for a flat FLRW spacetime,  $f(R)$ -gravity (without a matter source) provide de Sitter solutions, that is,  $R = R_0$ , when the following expression holds

$$R_0 f'(R_0) - 2f(R_0) = 0. \quad (45)$$

Note that from the power law models  $f(R) = R^n$ , only the quadratic model,  $n = 2$ , satisfies identically the condition (45). Moreover, since  $f' = \phi$ , and  $V_{,\phi} = R$ , expression (45) can be written as follows

$$2V(\phi_0) - \phi_0 V_{,\phi}|_{\phi=\phi_0} = 0 \quad (46)$$

where  $\phi_0 = f'(R_0)$ . We apply the last condition to find de Sitter solutions for the models which followed from the application of the group invariants.

For the potential  $V_I(\phi)$ , the application of (46) gives the that  $(\phi_0)^2 = (V_1/V_2)$ , while from the potential  $V_{II}(\phi)$ , we find that the only real solution  $(\phi_0)^2 = \sqrt{3}(V_2/V_1)^{1/4}$ . Furthermore for  $V_{III}(\phi)$ , from (46) we have that it provides de Sitter solution for  $\phi_0 = \frac{25}{16}(V_2/V_1)^2$ , whereas the  $V_{IV}(\phi)$ -model gives a de Sitter solution for  $\phi_0 = -\frac{1}{2}(V_1/V_2)$ . Here we would like to remark that condition (46) for the  $V_I(\phi)$  and  $V_{IV}(\phi)$  models holds and for  $\phi_0 = 0$ , however that leads to not physical accepted solutions.

Finally for the  $f_V(R)$ -model, i.e. potential  $V_V(\phi)$ , the application of condition (46) gives four points which are

$$(\phi_0)^2 = \beta, \quad \phi_0 = -\frac{V_1 \mp \sqrt{V_1^2 - 16\beta(V_2)^2}}{4V_2}. \quad (47)$$

In the de Sitter solution,  $a(t) = \exp(H_0 t)$ , from (4) we have that  $R_0 = 12H_0^2$ . Hence by using the relation  $V_{,\phi}|_{\phi=\phi_0} = R_0$ , we can derive the value of the Ricciscalar. That provides us with information about the possible values of the free parameters of the models.

For the  $f_I(R)$ , and  $f_{II}(R)$  models, we find that  $R_0 \simeq V_1$ , which indicates that  $V_1 \simeq H_0^2$ . For the third model, namely  $f_{III}(R)$ , we find that  $R_0 = \frac{32}{125}(V_1)^3(V_2)^{-2}$ , that is,  $(V_1)^3 = \frac{375}{8}(V_2 H_0)^2$ , while for the  $f_{IV}(R)$ -model we have that in the de Sitter point  $(V_1)^3 = 48(V_2 H_0)^2$ .

Finally for the  $f_V(R)$  model, which provides us with four possible points with a de Sitter expansion the corresponding values of the Ricciscalar are

$$R_0 = 4\beta \left( V_1 \mp 4V_2 \sqrt{\beta} \right), \quad (48)$$

for  $\phi_0 = \pm\sqrt{\beta}$ , respectively and,

$$R_0 = \frac{V_1}{8V_2^2} (V_1^2 - 16V_2^2\beta) \mp \frac{(V_1^2 - 16V_2^2\beta)^{\frac{3}{2}}}{8V_2^2} \quad (49)$$

for the last two points. Since  $R_0 > 0$ , in the de Sitter point, expressions (48), (49) provide us with constraints for the possible physical accepted values of the free parameters of the model.

In the following Section we use the extra conservation laws in order to reduce the order of the field equations for the three new integrable models,  $V_{III}(\phi)$ ,  $V_{IV}(\phi)$  and  $V_V(\phi)$ . We do that by using the Hamilton-Jacobi theory.

## 5. SOLUTIONS OF THE FIELD EQUATIONS

In the coordinate system  $\{a, \phi\}$ , the Lagrangian (14) of the field equations is as follows

$$L(a, \dot{a}, \phi, \dot{\phi}) = 6a\phi\dot{a}^2 + 6a^2\dot{a}\dot{\phi} + a^3V(\phi). \quad (50)$$

From the last expression we define the momenta,  $p_a = \frac{\partial L}{\partial \dot{a}}$ ,  $p_\phi = \frac{\partial L}{\partial \dot{\phi}}$ , as follows

$$p_a = 12a\phi\dot{a} + 6a^2\dot{\phi}, \quad p_\phi = 6a^2\dot{a}. \quad (51)$$

The Hamiltonian, (15), in terms of the momentum has the following expression

$$\mathcal{E} = \frac{1}{6a^2} \left( p_a p_\phi - \frac{\phi}{a} p_\phi^2 \right) - a^3 V(\phi). \quad (52)$$

Moreover the field equations, (4) and (7), are equivalent to the following Hamiltonian system

$$\dot{a} = \frac{1}{6a^2} p_\phi, \quad \dot{\phi} = \frac{1}{6a^2} p_a - \frac{\phi}{3a^3} p_\phi, \quad (53)$$

$$\dot{p}_a = \frac{p_a p_\phi}{3a^3} - \frac{\phi}{2a^4} p_\phi^2 + 3a^2 V(\phi) \quad \text{and} \quad (54)$$

$$\dot{p}_\phi = \frac{p_\phi^2}{6a^3} + a^3 V_{,\phi}. \quad (55)$$

From the Hamiltonian (52) we define the Hamilton-Jacobi Equation

$$\frac{1}{6a^2} \left( \left( \frac{\partial S}{\partial a} \right) \left( \frac{\partial S}{\partial \phi} \right) - \frac{\phi}{a} \left( \frac{\partial S}{\partial \phi} \right)^2 \right) - a^3 V(\phi) - \left( \frac{\partial S}{\partial t} \right) = 0 \quad (56)$$

in which  $p_a = \frac{\partial S}{\partial a}$ ,  $p_\phi = \frac{\partial S}{\partial \phi}$ , and  $S = S(t, a, \phi)$ . Equation (56) provides us with the action  $S$ , which help us to reduce the dimension of the Hamiltonian system (53), (54). In the following we use the classification of Darboux [100] for the integrable systems in a two-dimensional manifold by following the notation of [101].

Before we proceed, as a final remark we would like to express the EoS which corresponds to the  $f(R)$  terms in the coordinates,  $\{a, \phi\}$ . From expression (12) we have that

$$w_f = -\frac{4H\dot{\phi} + 2\ddot{\phi} - V(\phi)}{6H\dot{\phi} - V(\phi)}. \quad (57)$$

However, from (50) we calculate the ‘‘Klein-Gordon’’ equation for the field  $\phi$ , namely

$$2\ddot{\phi} + 4H\dot{\phi} - 2\phi H^2 - V(\phi) + \frac{1}{3}\phi V_{,\phi} = 0 \quad (58)$$

by replacing  $\ddot{\phi}$  in (57) from (58). We find

$$w_f = -\frac{6\phi H^2 - 2\phi V_{,\phi}}{18H\dot{\phi} - 3V(\phi)}, \quad (59)$$

that is, the parameter in the EoS is expressed only in terms of the first derivatives of  $\{a, \phi\}$ .

### 5.1. $f_{III}(R)$ -model

For the  $f_{III}(R)$ -model with effective potential  $V_{eff} = a^3 V_{III}(\phi)$ , we define the new coordinates

$$a = \sqrt{u}, \quad \phi = \frac{1}{6} \frac{v^2}{\sqrt{u}} \quad (60)$$

in which the Hamiltonian (52) becomes

$$\mathcal{E} = \frac{p_u p_v}{v} - \left( V_1 u^{\frac{3}{2}} - \sqrt{6} V_2 \frac{u^{\frac{7}{4}}}{v} \right). \quad (61)$$

In the new coordinates the conservation law (33) has the following expression

$$I_{III} = p_v^2 - 2\frac{u}{v} p_u p_v + \frac{6}{5} V_1 u^{\frac{5}{2}} - 2\sqrt{6} V_2 \frac{u^{\frac{11}{4}}}{v}, \quad (62)$$

that is, the field equations form an integrable dynamical system where the supporting manifold is a Lie surface [101, 102]. We recall that another cosmological model, which is integrable and for which the supporting manifold is a Lie surface, is a specific case of the early dark energy model of a minimally coupled scalar field, for details see [66].

Therefore we have that the action,  $S$ , in the new coordinates,  $\{u, v\}$ , has the following form

$$S(t, u, v) = -\varepsilon \left( v S_0(u) + \int \frac{\sqrt{6} V_2 u^{\frac{7}{4}}}{S_0(u)} du \right) - \mathcal{E} t, \quad (63)$$

where  $S_0(u) = \sqrt{2\mathcal{E}u + I_{III} + \frac{4}{5}V_1 u^{\frac{5}{2}}}$ , and  $\varepsilon = \pm 1$ .

With the use of (63) the field equations are reduced to the following two first-order ordinary differential equations

$$v\dot{u} = -\varepsilon S_0(u), \quad (64)$$

and

$$v\dot{v} = \varepsilon \left( -v \frac{\mathcal{E} + V_1 u^{\frac{3}{2}}}{S_0(u)} + \frac{\sqrt{6} V_2 u^{\frac{7}{4}}}{S_0(u)} \right). \quad (65)$$

Dynamical systems supported by a Lie surface cannot necessarily be solved by the method of separation of variables. However, the importance of the existence of the Lie surface is that we can solve the reduced system and express the one dependent variable in terms of the other. From (64) we have  $\frac{dv}{dt} = -\frac{\varepsilon S_0(u)}{v} \frac{dv}{du}$ . Hence equation (65), becomes,

$$\frac{dv}{du} = v \frac{\mathcal{E} + V_1 u^{\frac{3}{2}}}{(S_0(u))^2} - \frac{\sqrt{6} V_2 u^{\frac{7}{4}}}{(S_0(u))^2}. \quad (66)$$

The solution of the latter is

$$v(u) = \left[ \int B(u) e^{-\int A(u) du} du + v_0 \right] e^{\int A(u) du}, \quad (67)$$

where

$$A(u) = \frac{\mathcal{E} + V_1 u^{\frac{3}{2}}}{(S_0(u))^2}, \quad B(u) = -\frac{\sqrt{6} V_2 u^{\frac{7}{4}}}{(S_0(u))^2}. \quad (68)$$

For instance, when  $I_3 = 0$  and  $\mathcal{E} = 0$ , the closed-form solution of (67) in terms of the scale factor is<sup>9</sup>

$$v(a) = v_1 a^{\frac{1}{2}} + v_0 a^{\frac{5}{2}}, \quad (69)$$

where,  $v_1 = \frac{5\sqrt{6}}{4} \frac{V_2}{V_1}$ . Hence from (60) for the field,  $\phi = f'(R)$ , we have the following expression,

$$\phi(a) = \frac{1}{6} (v_1 + 2v_0 v_1 a^2 + v_0^2 a^4). \quad (70)$$

From (60) and (64) the Hubble function, (67), is

$$H(a) = \frac{1}{2} \frac{S_0(u)}{u} \left[ \int B(u) e^{-\int A(u) du} du + v_0 \right]^{-1} e^{-\int A(u) du}, \quad (71)$$

in which for the solution, (69), the Hubble function is

$$H(a) = \sqrt{\frac{4}{5} V_1} a^{-\frac{1}{4}} (v_1 + v_2 a^2)^{-1}. \quad (72)$$

In order to study the behavior of the cosmological fluid, in fig. 5.1 we give the evolution for the EoS parameter (59) which follows from the solution (67) for the  $f_{III}(R)$ -model. We observe that the effective perfect fluid which follows from the terms which arise from the modified Friedmann's equations has an EoS parameter  $w_f \leq \frac{1}{3}$  in which can cross the phantom-divide line,  $w_f < -1$ , in the late universe for different values of the free parameters. The values of the free parameters  $V_1, V_2$  has been chosen such as to approximate the condition which follow from section 4.2 and the solution to give a de Sitter universe. Note that the Hubble constants which are provided by the models are  $H_0 \simeq 69.6 \text{ km/s/Mpc}$ .

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<sup>9</sup> We note that for  $I_3 = 0$  and  $V_1 = 0$  the closed-form solution of (67) is expressed in terms of the Whittaker Function.

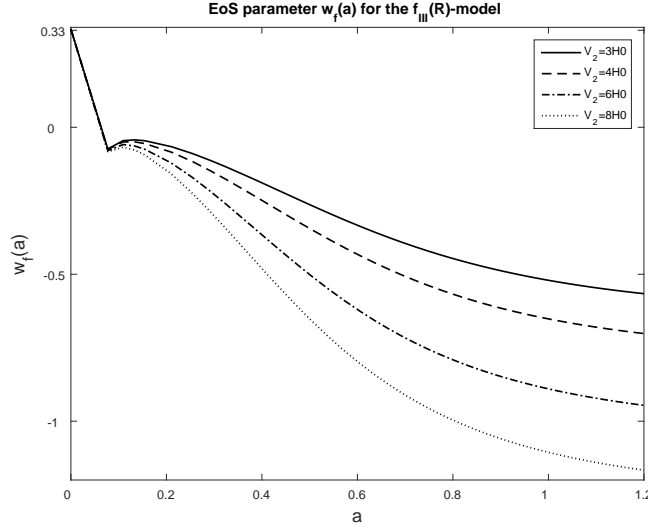


FIG. 2: Evolution of the EoS parameter  $w_f(a)$  for the  $f_{III}(R)$ -model. We observe that in the early universe the fluid which follow from  $f(R)$ -gravity act like a radiation fluid, i.e.,  $w_f(a) \simeq 1/3$  (see [35]). Moreover at the late universe the EoS parameter can cross the phantom-barrier and has a linear behavior which reach the deSitter point. For the plots we select:  $\mathcal{E} = 6 \Omega_{m0} H_0^2$ ,  $I_{III} = 0.2 H_0^2$ ,  $V_1 = H_0 (V_2 H_0)^{\frac{2}{3}}$ ,  $\varepsilon v(0) = 0.15$ ,  $\Omega_{m0} = 0.28$ , in units where  $H_0 = 69.6 \text{ km/s/Mpc}$ . The solid line is for  $V_2 = 3H_0$ , the dash-dot line is for  $V_2 = 4H_0$ , the dashed line is for  $V_2 = 6H_0$ , whereas the dotted line is for  $V_2 = 8H_0$ .

## 5.2. $f_{IV}(R)$ -model

For the  $f_{IV}(R)$ -model we define the new variable  $w = 6a\phi$ . Hence the Hamiltonian, (52), and the quadratic conservation law, (34), are written as follows,

$$\mathcal{E} = \frac{p_a p_w}{a} - \left( \bar{V}_1 w^3 + \bar{V}_2 \frac{w^4}{a} \right), \quad (73)$$

$$I_{IV} = p_a^2 - 2 \frac{w}{a} p_a p_w + \frac{3}{2} \bar{V}_1 w^4 + 2 \bar{V}_2 \frac{w^5}{a}, \quad (74)$$

where  $M_1 = 6^{-3} V_1$  and  $M_2 = 6^{-4} V_2$ . Easily we can see that the  $f_{IV}$ -model is integrable in which the supporting manifold is a Lie surface, similarly to the  $f_{III}$ -model.

From (73) with the use of (74) we find that the solution of the Hamilton-Jacobi equation is

$$S(t, a, w) = \varepsilon \left( \frac{a}{2} S_1(w) - 2 \int \frac{\bar{V}_2 w^4}{S_1(w)} \right) - \mathcal{E} t, \quad (75)$$

where  $\varepsilon = \pm 1$  and  $S_1(w) = \sqrt{2M_1 w^4 + 8\mathcal{E} w + 4I_{IV}}$

Therefore the field equations are reduced to the following system

$$a\dot{a} = -2\varepsilon \frac{a(\mathcal{E} + M_1 w^3) + M_2 w^4}{S_1(w)}, \quad (76)$$

and

$$a\dot{w} = -\frac{\varepsilon}{2} S_1(w). \quad (77)$$

Moreover in the limit for which,  $\mathcal{E} = 0$ ,  $I_{IV} = 0$ , and  $M_2 = 0$ , the closed-form solution of the scale factor is  $a(t) = a_0 t^2$ , in which we have applied the initial condition  $a(t \rightarrow 0) = 0$ . That solution corresponds to  $f_{IV}^1(R) \simeq R^3$  model and we can see that the solution describes a universe with a perfect fluid in which the parameter in the EoS is  $w_f = -\frac{1}{3}$ .

Similarly with the  $f_{III}(R)$ -model we can solve the one dependent parameter of the system (76), (77) in terms of the other. However, for the  $f_{III}(R)$ -model we did that by expressing the new variable,  $v$ , in terms of the scale factor. In this model we can express the scale factor in terms of the new variable,  $w$ , i.e.  $a(w)$ .

Hence with the use of (77) equation (76) becomes

$$\frac{da}{dw} = \bar{A}(w) a + \bar{B}(w), \quad (78)$$

where

$$\bar{A}(w) = \frac{4(\mathcal{E} + M_1 w^3)}{(S_1(w))^2}, \quad \bar{B}(w) = \frac{4M_2 w^4}{(S_1(w))^2}, \quad (79)$$

that is, the solution of  $a(w)$  is given by the formula (67).

We continue to the analysis of the last integrable model, namely  $f_V(R)$ -model.

### 5.3. $f_V(R)$ -model

For the last model which is given by the application of Killing tensors, contact symmetries, to the Lagrangian of the field equations we can see that the supporting manifold of the dynamical system is a Liouville surface. Hence the Hamilton-Jacobi Equation can be solved with the method of separation of variables.

Under the coordinate transformation

$$a = x + y, \quad \phi = \sqrt{\beta} \frac{x - y}{x + y} \quad (80)$$

the Hamiltonian function (52) is

$$\mathcal{E} = \frac{\frac{1}{2}p_x^2 - \frac{1}{2}p_y^2 - U_1 x^4 - U_2 y^4}{12\sqrt{\beta}(x + y)}, \quad (81)$$

where the constants  $U_1$  and  $U_2$ , are  $U_1 = 12\beta^2(V_1 + 4V_2\sqrt{\beta})$  and  $U_2 = -12\beta^2(V_1 - 4V_2\sqrt{\beta})$ . We see that, when  $V_1^2 = 16\beta V_2^2$ , we have that  $U_1 = 0$  or  $U_2 = 0$ .

Furthermore the quadratic conversation law (35) in the new coordinates has the following form

$$I_V = \frac{yp_x^2 + xp_y^2 - 2(U_1 y x^4 - U_2 x y^4)}{x + y}. \quad (82)$$

Therefore the solution of the Hamilton-Jacobi Equation is

$$S(t, x, y) = \varepsilon_1 \int \sqrt{I_5 - 24\sqrt{\beta}\mathcal{E}x + 2U_1 x^4} dx \\ + \varepsilon_2 \int \sqrt{I_5 + 24\sqrt{\beta}\mathcal{E}y - 2U_2 y^4} dy, \quad (83)$$

whereas the field equations reduce to the following system of first-order ordinary differential equations

$$12\sqrt{\beta}(x + y)\dot{x} = \varepsilon_1 \sqrt{I_5 - 24\sqrt{\beta}\mathcal{E}x + 2U_1 x^4}, \quad (84)$$

and

$$12\sqrt{\beta}(x + y)\dot{y} = -\varepsilon_2 \sqrt{I_5 + 24\sqrt{\beta}\mathcal{E}y - 2U_2 y^4}, \quad (85)$$

where  $\varepsilon_{1,2} = \pm 1$ .

Under the change of variable,  $dt = \frac{1}{a}d\tau$ , the closed-form solution of the system (84), (85) is given in terms of Elliptic functions.

From (80) with the use of (84) and (85) we derive the Hubble function

$$12\sqrt{\beta}H(a) = \varepsilon_1 \sqrt{I_5 - 24\sqrt{\beta}\mathcal{E}x + 2U_1 x^4} + \quad (86)$$

$$- \varepsilon_2 \sqrt{I_5 + 24\sqrt{\beta}\mathcal{E}y - 2U_2 y^4}, \quad (87)$$

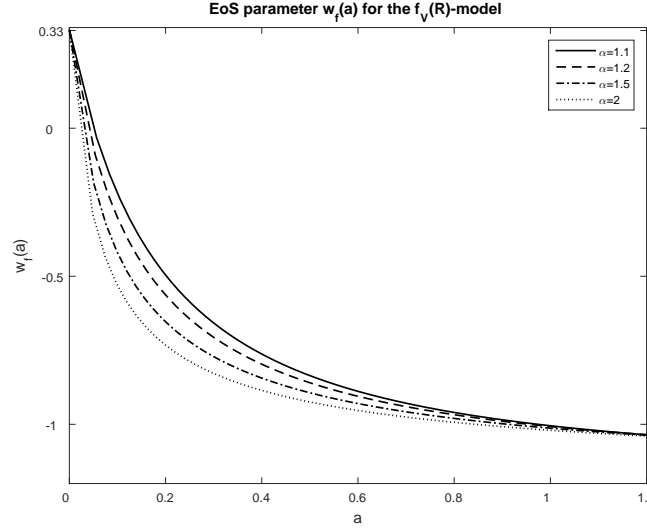


FIG. 3: Evolution of the parameter in the EoS,  $w_f(a)$ , for the  $f_V(R)$ -model. For the numerical solutions we select  $\mathcal{E} = 6\Omega_{m0}H_0^2$ ,  $I_{IV} = 0$ ,  $V_2 = H_0^{-1}$ ,  $V_1 = 4\alpha$ ,  $\varepsilon_{1,2} = +1$ ,  $a(0) = 0$  and  $\Omega_{m0} = 0.28$  and  $H_0 = 69.6 \text{ Km/s/Mpc}$ . The solid line is for  $\alpha = 1.1$ , the dash-dash line is for  $\alpha = 1.2$ , the dash-dot line is for  $\alpha = 1.5$  and the dot-dot line is for  $\alpha = 2$ .

In the limit for which  $\dot{x} \simeq 0$ ,  $x + y \simeq y$ ,  $\bar{U}_2 < 0$ , that is,  $V_1 > 4V_2\sqrt{\beta}$ , the latter can takes the following form

$$\left(\frac{H(a)}{H_0}\right)^2 \simeq \Omega_{r0}a^{-4} + \Omega_{m0}a^{-3} + \Omega_\Lambda \quad (88)$$

in which  $I_5 = 144\beta\Omega_{r0}H_0^2$ , and  $\Omega_\Lambda \simeq U_2$ . The last equation describe a universe in General Relativity with cosmological constant, dark matter and radiation fluid. We can see that the density of the radiation term, which is provided by the  $f(R)$ -theory is related with the value of the Noetherian conservation law  $I_5$ . However this is only a particular solution and the general behavior of the Hubble function is different.

From the system (84), (85) we observe that the free parameters of the model that we have to determine are,  $\mathcal{E} = 6\Omega_{m0}H_0^2$ ,  $\beta$ ,  $V_1$ ,  $V_2$ , the value of the conservation law  $I_5$ , the initial conditions  $(x_0, y_0) = (x, y)|_{t \rightarrow t_0}$ , and  $\varepsilon_1$ ,  $\varepsilon_2$ . In order to reduce the number of the free parameters we apply the initial condition  $a(t \rightarrow 0) \simeq 0^+$ , which gives that  $x_0 \simeq y_0$ . Moreover in the de Sitter points, in which the Ricciscalar is given by the expressions (48) and (49), we observe that if we set,  $\beta \simeq H_0^2$ ,  $\beta V_1 \simeq H_0^2$ , and  $\beta V_2 \simeq H_0$ , then  $R_0 \simeq H_0^2$ . As we discussed the value of the conservation law  $I_5$  can be related with the energy density of the radiation fluid which is introduced by the theory, in the present era  $\Omega_r$  is small, that is, we can assume that  $I_5 \simeq 0$ .

Furthermore we select that  $\beta = H_0^2$ ,  $V_2 = \gamma H_0^{-1}$  and  $V_1 = 4\alpha\gamma$ , with  $\alpha > 1$ ,  $\gamma > 0$ , and the initial conditions  $(x_0, y_0)$  are that in order the present value of the Hubble constant to be  $H_0 = 69.6 \text{ km/s/Mpc}$  [103]. Finally we choose the solution in which  $\varepsilon_{1,2} = +1$ , and the free parameters of the problem to be  $\{\alpha, \gamma, \Omega_{m0}\}$ .

For  $\gamma = 1$ ,  $\Omega_{m0} = 0.28$ , and for  $\alpha \in (1, 2)$ , in Fig. 3 the numerical evolution for the equation of state parameter  $w_f$ , for the  $f_V(R)$ -model is given. From the Fig. we observe that  $w_f \leq 1/3$  and  $w_f$  can cross the phantom barrier. Specifically, the EoS parameter  $w_f$ , it decreases rapidly and takes a negative value. Then the rate of decrease becomes slower where it has a linear behavior of the form  $w_f(a) = w_1a + w_0$ ,  $w_1 < 0$ , which is the CPL parametric dark energy model introduced by Chevallier, Polarski [104], and Linder [105].

In order to test the viability of the  $f_V(R)$ -model we perform a joint likelihood analysis using the Type Ia supernova data set of Union 2.1 [106], and the BAO data [107, 108] in which we select the free parameters of the model to be  $\{\alpha, \gamma, \Omega_{m0}\}$ . We fit the model with the data using the gradient-search method [109], for different set of random numbers in the space of the free parameters in order to avoid local minimum in the chi-square space, for the free parameters we select the range  $\alpha \in (1, 2)$ ,  $\gamma \in (0.5, 1.2)$ , and  $\Omega_{m0} = (0.25, 0.35)$ . We find that the best fit parameters are  $(\alpha, \gamma, \Omega_{m0})_{fit} = (1.24, 0.8, 0.28)$ , in which the  $(\chi_{\min}^2)_{total}^{fV} = \min(\chi_{SNIa}^2 + \chi_{BAO}^2) \simeq 559$ , while with the same algorithm, for the  $\Lambda$ -cosmology<sup>10</sup> we find that  $(\chi_{\min}^2)_{total}^\Lambda = \min(\chi_{SNIa}^2 + \chi_{BAO}^2) \simeq 560$ . The small difference on the

<sup>10</sup> Recall that in  $\Lambda$ -cosmology, the Hubble function is  $H(z) = H_0 \left[ (1 - \Omega_{m0}) + \Omega_{m0}(1+z)^3 \right]^{1/2}$ .

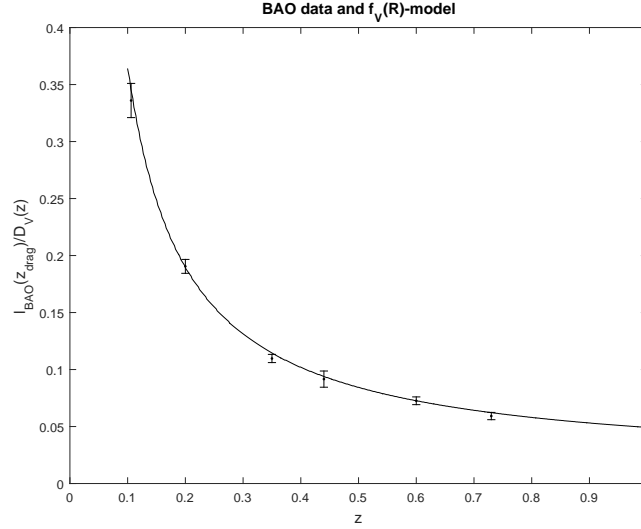


FIG. 4: BAO data and evolution of the  $d_{th} = l_{BAO}(z_{drag})(D_V(z))^{-1}$ , parameter for the  $f_V(R)$ -model for the parameters  $(\alpha, \gamma, \Omega_{m0})_{fit} = (1.24, 0.8, 0.28)$  in which  $(\chi^2_{min})_{total}^{f_V} \simeq 559$ .

minimum chi-square value between the two models,  $|\Delta\chi^2_{min}| \simeq 1$ , indicates that the two models fit the data with a similar way. In Fig. 4 we plot the theoretical parameter<sup>11</sup>  $d_{zth} = l_{BAO}(z_{drag})(D_V(z))^{-1}$ , for the  $f_V(R)$ -model and the six BAO data with the corresponding errors [108].

## 6. CONCLUSIONS

In this work we considered an FLRW spacetime in which the gravitational Action Integral is that of  $f(R)$ -gravity with a dustlike fluid. In order to determine the functional form of the  $f(R)$  function we applied as selection rule the existence of Noether symmetries for the field equations which are followed by Lie-Bäcklund transformations linear in the momentum, contact transformations. The importance of this kind of symmetries is that they provide us with quadratic conservation laws.

As the Lagrangian of the field equations is in the form of classical physical dynamical systems of the form  $L(x^k, \dot{x}^k) = T - V$ , where  $T$  is the kinetic energy and  $V$  the potential, we were able to apply the existence results in the literature in order to perform our classification. Hence contact symmetries if  $f(R)$ -gravity is generated by the Killing tensors of the minisuperspace, that is, the theory. In fact the Killing tensors of the minisuperspace select the model.

For a spatially flat universe we found five models which admit quadratic conservation laws, where for spatially nonflat universe we found only three models which are included in the five models of the spatially flat FLRW spacetime. From the five models, two models are well known in the literature and they have been found from the application of Noether's Theorem for point transformations. Moreover the quadratic conservation laws which correspond to the five models are in involution with the Hamiltonian function, that is, the field equations are integrable.

For the three new models with the use of the extra quadratic conservation laws we reduced the field equations to a system of two nonlinear first-order ordinary differential equations. We performed numerical simulations for the models and we studied the evolution of the parameter in the EoS for the fluid components which corresponds to the  $f(R)$ -theory. For all the models we show that the parameter in the EoS has an upper bound which is  $w_f \leq 1/3$ . There is no lower bound which means that  $w_f$  can cross the phantom divide line. Furthermore, for the  $f_{III}(R)$  and the  $f_V(R)$  models the cosmological fluid which follows from the additional terms of the Friedmann equations in the present time the parameter in the EoS has a linear behavior given by the linear function  $w_f(a) = w_1 a + w_0$ . Furthermore, for the  $f_V(R)$ -model, and from (86), we showed that the value of the second conservation law is related to the fluid components which are introduced by  $f(R)$ -gravity and, specifically, it is related to the density of the

<sup>11</sup> Where  $l_{BAO}(z_{drag})$  is the BAO scale at the drag redshift, and  $D_V(z)$  is the volume distance



radiation fluid. That is an important result which indicates a relation between the conservation laws and physical observable quantities. However, the exact physical properties of the conservation laws of the field equations are still unknown. Furthermore for the  $f_V(R)$ -model we showed that can fit the cosmological data with a similar way with the  $\Lambda$ -cosmology.

Another issue that we did not discuss in this work is the connection of these  $f(R)$ -models with other conformally equivalent theories. The reason for which we restricted our analysis is because we considered the dustlike fluid which does not interacting with gravity. Of course if we relaxed that restriction or there is no dustlike fluid, i.e.,  $\mathcal{E} = 0$ , then the solutions we have found also hold and for conformally equivalent theories. However, in that case someone should extend the application of contact transformations which leave invariant the field equations not only to that which follow from the Killing tensors but also to the Conformal Killing tensors. For a discussion on the relation of symmetries and conservation laws of conformal equivalence theories see [110].

This work extends the analysis of group invariant transformations in gravitational physics and cosmology and shows that the application of group invariants in modified theories provides us with models which can describe the late-time acceleration phase of the universe.

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### Appendix A: Quadratic conservation laws in spatially nonflat $f(R)$ -models

In this Appendix we complete our analysis on the  $f(R)$ -models which admit quadratic conservation laws if FLRW spacetime has nonvanishing spatial curvature. Above we considered that the FLRW had zero spatial curvature. In the case where the spatial curvature is  $K$ , with  $K \neq 0$ , from the Action Integral, (2), and with the use of the Lagrange Multiplier we find the following Lagrangian for the field equations

$$L(a, \dot{a}, \phi, \dot{\phi}) = 6a\phi\dot{a}^2 + 6a^2\dot{a}\dot{\phi} + a^3V(\phi) - 6Ka\phi. \quad (\text{A1})$$

We can see that this Lagrangian admits the same minisuperspace as the Lagrangian, (50). Therefore in order to apply the method of Section (3) for the existence of a contact transformation which leaves the Action Integral, (2), invariant with Lagrangian, (A1), we use the KTs of Appendix B.

Hence, when  $V(\phi) = V_I(\phi)$ , the modified field equations admit the conservation law

$$\bar{I}_I = I_I(a, \dot{a}, \phi, \dot{\phi}). \quad (\text{A2})$$

Moreover, in the case for which  $V(\phi) = V_{IV}(\phi)$ , the quadratic conservation law of the field equations is

$$\bar{I}_{IV} = I_{IV}(a, \dot{a}, \phi, \dot{\phi}) - 6K(\alpha\phi)^2, \quad (\text{A3})$$

whereas for  $V_V(\phi)$  the quadratic conservation law is

$$\bar{I}_V = I_V(a, \dot{a}, \phi, \dot{\phi}) - 3Ka^2(\phi^2 - \beta). \quad (\text{A4})$$

Functions  $I_I$ ,  $I_{IV}$  and  $I_V$  are given by the expressions (31), (34) and (35), respectively.

### Appendix B: Killing vectors and Killing tensors

The minisuperspace, (25), is the  $M^2$  space, which means that it admits a three-dimensional Killing algebra. The KVs in the coordinates  $\{a, \phi\}$  are:

$$K_1^i = a\partial_a - 3\phi\partial_\phi, \quad K_3^i = \frac{1}{a}\partial_a \quad \text{and} \quad (\text{B1})$$

$$K_2^i = \frac{1}{a} \left( \partial_a - \frac{\phi}{a} \right) \partial_a. \quad (\text{B2})$$

Moreover,  $M^2$ , admits five KTs (except the metric tensor  $\gamma_{ij}$ ), which are of the form  $K_{AB}^{ij} = K_A^{(i} \otimes K_B^{j)}$ , where  $A, B = 1, 2, 3$ . Hence the five KTs in the coordinates  $\{a, \phi\}$  are

$$K_{11}^{ij} = \begin{pmatrix} a^2 & -3a\phi \\ -3a\phi & 9\phi^2 \end{pmatrix}, \quad K_{22}^{ij} = \begin{pmatrix} 1/a^2 & -\phi/a^3 \\ -\phi/a^3 & \phi^2/a^4 \end{pmatrix}, \quad (\text{B3})$$

$$K_{33}^{ij} = \begin{pmatrix} 0 & 0 \\ 0 & a^{-2} \end{pmatrix}, \quad K_{12}^{ij} = \begin{pmatrix} 1 & -2a/\phi \\ -\phi/a^3 & 3\phi^2/a^2 \end{pmatrix} \quad \text{and} \quad (\text{B4})$$

and

$$K_{13}^{ij} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & -6\phi/a^2 \end{pmatrix}. \quad (\text{B5})$$

Recall that  $K_{23}^{ij} = K_2^{(i} \otimes K_3^{j)}$  is the metric tensor  $\gamma^{ij}$ .

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